Probability Supplement

Atoms

Let $\phi(x)$ denote the standard normal density function and $\Phi(x)$ the standard normal distribution function. An example of a distribution with an atom is

$$F(x) = \begin{cases} 0.8\Phi(x) & -\infty < x < 0 \\ 0.2 & x = 0 \\ 0.8\Phi(x) & 0 < x < \infty \end{cases}$$

If $g(x)$ is a bounded, continuous function its expectation is computed as

$$\mathcal{E}g = 0.8 \int_{(-\infty,0)} g(x)\phi(x)\,dx + 0.2g(0) + 0.8 \int_{(0,\infty)} g(x)\phi(x)\,dx$$

or, because $\Phi(x)$ is a continuous distribution,

$$\mathcal{E}g = 0.8 \int_{-\infty}^{\infty} g(x)\phi(x)\,dx + 0.2g(0)$$

Chapman-Kolmogorov equation

Let $p(x,y)$ be the transition density of a Markov process, i.e., the conditional density of y given x. One way to prove that $f(x)$ is the stationary density implied by the transition density $p(x,y)$ is to show that it satisfies the Chapman-Kolmogorov equation

$$\int p(x,y)f(x)\,dx = f(y)$$

Equivalently one can show that

$$\int g(y) \int p(x,y)f(x)\,dx\,dy = \int g(y)\,f(y)\,dy$$

for every function $g(x)$ that is bounded and continuous. However, by interchanging the order of integration on the left, one can see that this is the same as showing that

$$\int \mathcal{E}(g|x)\,f(x)\,dx = \int g(y)\,f(y)\,dy$$

The advantage of the last expression is that it is more convenient to manipulate than the others when a distribution has atoms.