Bayesian Estimation of a Dynamic Game with Endogenous, Partially Observed, Serially Correlated State

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Outline

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  - ▷ Econometric Problem
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- Example
- Econometrics
- Simulation

Econometric Problem

- Estimate a dynamic game
  - ▷ with partially observed state
  - ▷ with serially correlated state
  - ▷ with (possibly) endogenous state
  - ▷ with (possibly) complete information
  - ▷ with continuous or discrete choice
  - ▷ with (mixed) continuous or discrete state
- Applications:
  - ▷ Entry and exit from industry, technology adoption, technology upgrades, introduction of new products, discontinuation of old products, relocation decisions, etc.

Econometric Approach

- Bayesian econometrics
  - ▷ accommodates a nondifferentiable, nonlinear likelihood
  - ▷ easy to parallelize
  - ▷ allows the use of prior information
- Develop a general solution algorithm
  - ▷ computes pure strategy subgame perfect Markov equilibria
  - ▷ using a locally linear value function
- Use sequential importance sampling (particle filter)
  - ▷ to integrate unobserved variables out of the likelihood
  - ▷ to estimate ex-post trajectory of unobserved variables

Results

- Method is exact
  - ▷ Stationary distribution of MCMC chain is the posterior.
  - ▷ Because we prove the computed likelihood is unbiased.
  - ▷ Efficient, number of required particles is small.
- Regularity conditions minimal.

Table 1. Related Literature

Game	One Player	Two or more Players
Static incomplete observed	$\sim$	17
Dynamic incomplete observed	$\sim$	9
Dynamic incomplete discrete unobserved	$\sim$	1
Static complete unobserved	1	5
Dynamic complete unobserved correlated	3	**0**

Outline

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- Simulation

Table 2. Generic pharmaceuticals, Scott-Morton (1999)

Drug / Active Ingredient	ANDA Date	Dominant Firms (enter = 1, not enter = 0) Mylan	Novopharm	Lemmon	Geneva	Total Entrants	Revenue ($'000s)
Sulindac	03 Apr. 90	1	0	1	1	7	189010
Erythromycin Stearate	15 May 90	0	0	0	0	1	13997
Atenolol	31 May 90	1	0	0	0	4	69802
Nifedipine	04 Jul. 90	0	1	0	0	5	302983
Minocycline Hydrochloride	14 Aug. 90	0	0	0	0	3	55491
Methotrexate Sodium	15 Oct. 90	1	0	0	0	3	24848
Pyridostigmine Bromide	27 Nov. 90	0	0	0	0	1	2113
Estropipate	27 Feb. 91	0	0	0	0	2	6820
Loperamide Hydrochloride	30 Aug. 91	1	1	1	1	5	31713
Phendimetrazine	30 Oct. 91	0	0	0	0	1	1269
Tolmetin Sodium	27 Nov. 91	1	1	1	1	7	59108
Clemastine Fumarate	31 Jan. 92	0	0	1	0	1	9077
Cinoxacin	28 Feb. 92	0	0	0	0	1	6281
Diltiazem Hydrochloride	30 Mar. 92	1	1	0	0	5	439125
Nortriptyline Hydrochloride	30 Mar. 92	1	0	0	1	3	187683
Triamterene	30 Apr. 92	0	0	0	1	2	22092
Piroxicam	29 May 92	1	1	1	0	9	309756
Griseofulvin Ultramicrocrystalline	30 Jun. 92	0	0	0	0	1	11727
Pyrazinamide	30 Jun. 92	0	0	0	0	1	306
Diflunisal	31 Jul. 92	0	0	1	0	2	96488
Carbidopa	28 Aug. 92	0	0	1	0	4	117233
Pindolol	03 Sep. 92	1	1	0	1	7	37648
Ketoprofen	22 Dec. 92	0	0	0	0	2	107047
Gemfibrozil	25 Jan. 93	1	0	1	0	5	330539
Benzonatate	29 Jan. 93	0	0	0	0	1	2597
Methadone Hydrochloride	15 Apr. 93	0	0	0	0	1	1858
Methazolamide	30 Jun. 93	0	0	0	1	3	4792
Alprazolam	19 Oct. 93	1	1	0	0	7	614593
Nadolol	31 Oct. 93	1	0	0	0	2	125379
Levonorgestrel	13 Dec. 93	0	0	0	0	1	47836
Metoprolol Tartrate	21 Dec. 93	1	1	0	1	9	235625
Naproxen	21 Dec. 93	1	1	1	1	8	456191
Naproxen Sodium	21 Dec. 93	1	1	1	1	7	164771
Guanabenz Acetate	28 Feb. 94	0	0	0	0	2	18120
Triazolam	25 Mar. 94	0	0	0	0	2	71282
Glipizide	10 May 94	1	0	0	0	1	189717
Cimetidine	17 May 94	1	1	0	0	3	547218
Flurbiprofen	20 Jun. 94	1	0	0	0	1	155329
Sulfadiazine	29 Jul. 94	0	0	0	0	1	72
Hydroxychloroquine Sulfate	30 Sep. 94	0	0	0	0	1	8492
Mean		0.45	0.28	0.25	0.25	3.3	126901

Entry Game Characteristics

- Heterogeneous unobservable costs
  - ▷ Serially correlated costs.
- Complete information
  - ▷ Firms know each other's revenue and costs.
- Endogenous state
  - ▷ Entry changes future costs.
    - ∗ Capacity constraint: increased costs.
    - ∗ Learning: decreased costs.
- Simultaneous move dynamic game.

An Entry Game I

- There are $i = 1, \ldots, I$, firms that are identical ex ante.

- Firms maximize PDV of profits over $t, \ldots, \infty$

- Each period t a market opens and firms make entry decisions:

  ▷ If enter $A_{i,t} = 1$, else $A_{i,t} = 0$.

- Number of firms in the market at time t, is $N_t = \sum_{i=1}^{I} A_{i,t}$.

An Entry Game II

- Gross revenue R_t is exogenously determined.
- A firm's payoff is $R_t/N_t - C_{i,t}$ where $C_{i,t}$ is "cost".
- Costs are endogenous to past entry decisions:
  - ▷ $c_{i,t} = c_{i,u,t} + c_{i,k,t}$ (lower case denotes logs)
  - ▷ $c_{i,u,t} = \mu_c + \rho_c\,(c_{i,u,t-1} - \mu_c) + \sigma_c e_{it}$
  - ▷ $c_{i,k,t} = \rho_a\, c_{i,k,t-1} + \kappa_a A_{i,t-1}$
  - ▷ Source of the dynamics
- Coordination game: If multiple equilibria (rare), the lowest cost firms are the entrants.

Solution I: Bellman Equation

For each player

$$
\begin{aligned}
& V_i(C_{it}, C_{-i,t}, R_t) \\
& \quad = \; A^E_{it} \left(R_t / N^E_t - C_{it} \right) \\
& \qquad + \beta \, \mathcal{E}[V_i(C_{i,t+1}, C_{-i,t+1}, R_{t+1}) \mid A^E_{i,t}, A^E_{-i,t}, C_{i,t}, C_{-i,t}, R_t]
\end{aligned}
$$

The value function for all players is

$$
V(C_t, R_t) = (V_1(C_{1t}, C_{-1t}, R_t), \ldots, V_I(C_{It}, C_{-It}, R_t))
$$

– $V(c_t, r_t)$ is approximated by a local linear function.

– The integral is computed by Gauss-Hermite quadrature.

Solution II: Subgame Perfect Markov Equilibrium

Equilibrium condition (Nash)

$$V_i(A^E_{i,t}, A^E_{-i,t}, C_{i,t}, C_{-i,t}, R_t) \geq V_i(A_{i,t}, A^E_{-i,t}, C_{i,t}, C_{-i,t}, R_t) \quad \forall\, i, t.$$

where

$$\begin{aligned} & V_i(A_{i,t}, A_{-i,t}, C_{i,t}, C_{-i,t}, R_t) \\ & \quad = A_{it}\,(R_t/N_t - C_{it}) \\ & \qquad + \beta \mathcal{E}[V_i(A^E_{i,t+1}, A^E_{-i,t+1}, C_{i,t+1}, C_{-i,t+1}, R_{t+1}) | A_{i,t}, A_{-i,t}, C_{i,t}, C_{-i,t}, R_t] \end{aligned}$$

is the choice-specific payoff function.

Complete information: C_t, R_t known implies A^E_t known whence

$$V_i(A^E_{i,t+1}, A^E_{-i,t+1}, C_{i,t+1}, C_{-i,t+1}, R_{t+1}) = V_i(C_{i,t+1}, C_{-i,t+1}, R_{t+1})$$

Solution III: Local Linear Approximation

- The value function V is approximated as follows:
  - ▷ Define a coarse grid on $s = (c_{u,1}, \ldots, c_{u,I}, r, c_{k,1}, \ldots, c_{k,I})$.Each hypercube of the grid is indexed its centroid K, called its key. The local linear approximation over the Kth hypercube is $V_K(s) = b_K + (B_K)s$.
  - ▷ For a three player game V_K is 3×1, b_K is 3×1 , B_K is 3×7, and s is 7×1.

- The local approximator is determined at key K by (1) solving the game at a set $\{s_j\}$ of states within the Kth hypercube, (2) computing $\{V_j = V(s_j)\}$ using the Bellman equation, and (3) computing the coefficients b_K and B_K by regressing $\{V_j\}$ on $\{s_j\}$. Continue until b_K and B_K stabilize.
  - ▷ Usually only 6 hypercubes are visited.

An Entry Game – Summary

- Log revenue: r_t

- Log costs: $c_{i,t} = c_{i,u,t} + c_{i,k,t} \qquad i = 1, \ldots, I$
  - ▷ $c_{i,u,t} = \mu_c + \rho_c\,(c_{i,u,t-1} - \mu_c) + \sigma_c e_{it}$
  - ▷ $c_{i,k,t} = \rho_a\, c_{i,k,t-1} + \kappa_a A_{i,t-1}$

- Parameters: $\theta = (\mu_c, \rho_c, \sigma_c, \mu_r, \sigma_r, \rho_a, \kappa_a, \beta, p_a)$

- Solution method: $A_t^E = S(c_{u,t}, c_{k,t}, r_t, \theta)$
  - ▷ A deterministic function.

Outcome Uncertainty

- Error density
  - ▷ $p(A_t \mid A_t^E, \theta) = \prod_{i=1}^{I} (p_a)^{\delta(A_{it}=A_{it}^E)} (1 - p_a)^{1-\delta(A_{it}=A_{it}^E)}$
  - ▷ $A_t^E = S(c_{u,t}, c_{k,t}, r_t, \theta)$
- Two scenerios
  - ▷ **Boundedly rational:** Ignore outcome uncertainty.
  - ▷ **Fully rational:** Take outcome uncertainty into account.
    - ∗ Bellman equations modified to include error density.

Abstraction

The state vector is

$$x_t = (x_{1t}, x_{2t}), \tag{1}$$

where x_{1t} is not observed and x_{2t} is observed. The observation (or measurement) density is

$$p(a_t \,|\, x_t, \theta). \tag{2}$$

The transition density is

$$p(x_t \,|\, a_{t-1}, x_{t-1}, \theta). \tag{3}$$

Its marginal is

$$p(x_{1t}|a_{t-1}, x_{t-1}, \theta). \tag{4}$$

The stationary density is

$$p(x_{1t} \,|\, \theta). \tag{5}$$

Assumptions

- We can draw from $p(x_{1t} \,|\, a_{t-1}, x_{t-1}, \theta)$ and $p(x_{1t} \,|\, \theta)$.
  - ▷ Can draw a sample from $p(x_{1t} \,|\, \theta)$ by simulating the game, and discarding a_t and x_{2t}.
  - ▷ Can draw from $p(x_{1,t} \,|\, a_{t-1}, x_{t-1}, \theta)$ by drawing from $p(x_t \,|\, a_{t-1}, x_{t-1}, \theta)$ and discarding x_{2t}.
- There is an analytic expression or algorithm to compute $p(a_t \,|\, x_t, \theta)$, $p(x_t \,|\, a_{t-1}, x_{t-1}, \theta)$, and $p(x_{1t}|a_{t-1}, x_{t-1}, \theta)$.
- If evaluating or drawing from $p(x_{1t}|a_{t-1}, x_{t-1}, \theta)$ is difficult some other importance sampler can be substituted.

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- Conclusion

Estimation Overview

1. In an MCMC loop, propose a parameter value **and a seed**.

2. Given the parameter value **and the seed**, compute an unbiased estimator of the integrated likelihood.

   - Integrate by evaluating a likelihood that includes latent variables at particles for those latent variables and averaging.

3. Use the estimate of the integrated likelihood to make the accept/reject decision of the MCMC algorithm.

The Likelihood

- With latent variables

$$L_t(\theta) = \left[\prod_{s=1}^{t} p(a_t \,|\, x_s, \theta)\, p(x_s \,|\, a_{s-1}, x_{s-1}, \theta) \right] p(a_0, x_0 \,|\, \theta)$$

- Without latent variables

$$\mathcal{L}(\theta) = \prod_{t=1}^{T} \int \cdots \int L_t(\theta) \prod_{s=0}^{t} dx_{1,s}$$

- Integrate by averaging sequentially over progressively longer particles. Concatenated draws for fixed k that start at time s and end at time t are denoted

$$\tilde{x}_{1,s:t}^{(k)} = (\tilde{x}_{1,s}^{(k)}, \ldots, \tilde{x}_{1,t}^{(k)});$$

  $\tilde{x}_{1,0:t}^{(k)}$ is called a particle.

Particle Filter

1. For $t = 0$

   (a) Start N particles by drawing $\tilde{x}_{1,0}^{(k)}$ from $p(x_{1,0} \,|\, \theta)$ using s as the initial seed and putting $\bar{w}_0^{(k)} = \frac{1}{N}$ for $k = 1, \ldots, N$.

   (b) If $p(a_t, x_{2t} \,|\, x_{1,t-1}, \theta)$ is available, then compute $\widehat{C}_0 = \frac{1}{N} \sum_{k=1}^{N} p(a_0, x_{2,0} \,|\, \tilde{x}_{1,0}^{(k)}, \theta)$ otherwise put $\widehat{C}_0 = 1$.

   (c) Set $x_{1,0:0}^{(k)} = \tilde{x}_{1,0}^{(k)}$, $x_{1,0}^{(k)} = x_{1,0:0}^{(k)}$, and $w_0^{(k)} = \frac{1}{N}$.

2. For $t = 1, \ldots, n$

   (a) For each particle, draw $\tilde{x}_{1t}^{(k)}$ from the transition density

   $$p(x_{1t} \,|\, a_{t-1}, x_{1,t-1}^{(k)}, x_{2,t-1}, \theta).$$

   (b) Compute

   $$\begin{aligned} \bar{v}_t^{(k)} &= \frac{p(a_t \,|\, \tilde{x}_{1,t}^{(k)}, x_{2,t}, \theta)\, p(\tilde{x}_{1,t}^{(k)}, x_{2,t} \,|\, a_{t-1}, x_{1,t-1}^{(k)}, x_{2,t-1}, \theta)}{p(\tilde{x}_{1,t}^{(k)} \,|\, a_{t-1}, x_{1,t-1}^{(k)}, x_{2,t-1}, \theta)} \\ \widehat{C}_t &= \frac{1}{N} \sum_{k=1}^{N} \bar{v}_t^{(k)} \end{aligned}$$

   (Note that the draw pair is $(\tilde{x}_t^{(k)}, x_{t-1}^{(k)})$ and the weight pair is $(\bar{v}_t^{(k)}, \frac{1}{N})$.)

(c) Set

$$\tilde{x}_{1,0:t}^{(k)} = \left(x_{1,0:t-1}^{(k)}, \tilde{x}_{1,t}^{(k)}\right).$$

(d) Compute the normalized weights

$$\hat{w}_t = \frac{\bar{v}_t^{(k)}}{\sum_{k=1}^{N} \bar{v}_t^{(k)}}$$

(e) For $k = 1, \ldots, N$ draw $x_{1,0:t}^{(k)}$ by sampling with replacement from the set $\{\tilde{x}_{1,0:t}^{(k)}\}$ according to the weights $\{\hat{w}_t^{(k)}\}$. (Note the convention: Particles with unequal weights are denoted by $\{\tilde{x}_{0:t}^{(k)}\}$. After resampling the particles are denoted by $\{x_{0:t}^{(k)}\}$.)

(f) Set $x_t^{(k)}$ to the last element of $x_{1,0:t}^{(k)}$.

3. Done

   (a) An unbiased estimate of the likelihood is

$$\ell' = \prod_{t=0}^{T} \widehat{C}_t$$

   and s' is the last seed returned in Step 2e.

Why Does This Work?

- For each particle, draw $\tilde{x}_{1t}^{(k)}$ from the transition density

$$p(x_{1t} \,|\, a_{t-1}, x_{1,t-1}^{(k)}, x_{2,t-1}, \theta).$$

- Compute

$$\begin{aligned}
\bar{v}_t^{(k)} &= \frac{p(a_t \,|\, \tilde{x}_{1,t}^{(k)}, x_{2,t}, \theta)\, p(\tilde{x}_{1,t}^{(k)}, x_{2,t} \,|\, a_{t-1}, x_{1,t-1}^{(k)}, x_{2,t-1}, \theta)}{p(\tilde{x}_{1,t}^{(k)} \,|\, a_{t-1}, x_{1,t-1}^{(k)}, x_{2,t-1}, \theta)} \\
\hat{C}_t &= \sum_{k=1}^{N} \bar{v}_t^{(k)} w_{t-1}^{(k)}
\end{aligned}$$

- An unbiased estimate of the likelihood is

$$\ell(\theta, s) = \prod_{t=0}^{T} \hat{C}_t$$

where s is the initial seed.

Verification Requires Some Notation

- In the Bayesian paradigm, θ and $\{a_t, x_t\}_{t=-\infty}^{\infty}$ are defined on a common probability space. Let $\mathcal{F}_t = \sigma\left\{\{a_s, x_{2s}\}_{s=-T_0}^{t}, \theta\right\}$.
- The elements of a_t and x_t may be either real or discrete. For z a vector with some coordinates real and the others discrete, let $\lambda(z)$ denote a product measure whose marginals are either counting measure or Lebesgue ordered to define an integral of the form $\int g(z)\, d\lambda(z)$.
- Particle filters are implemented by drawing independent uniform random variables $u_t^{(k)}$ and then evaluating a function* of the form $X_{1t}^{(k)}(u)$ and putting $\tilde{x}_{1t}^{(k)} = X_{1t}^{(k)}(u_t^{(k)})$ for $k = 1, \ldots, N$. Denote integration with respect to $(u_t^{(1)}, \ldots, u_t^{(N)})$ with $X_{1t}^{(k)}(u)$ substituted into the integrand by $\tilde{\mathcal{E}}_{1t}$. $\tilde{\mathcal{E}}_{1,0:t}$ is defined similarly.

*E.g., a conditional probability integral transformation.

To Show

Given weights $\bar{w}_t^{(k)}$, $k = 1, \ldots, N$, that satisfy

$$\prod_{s=0}^{t} C_s = \tilde{\mathcal{E}}_{1,0:t}\, \mathcal{E}\left[\sum_{k=1}^{N} \bar{w}_t^{(k)} \,|\, \mathcal{F}_t\right],$$

we seek to generate weights $\bar{w}_{t+1}^{(k)}$ that satisfy

$$\prod_{s=0}^{t+1} C_s = \tilde{\mathcal{E}}_{1,t+1} \tilde{\mathcal{E}}_{1,0:t}\, \mathcal{E}\left[\sum_{k=1}^{N} \bar{w}_{t+1}^{(k)} \,|\, \mathcal{F}_{t+1}\right],$$

where

$$\mathcal{L}(\theta) = \prod_{s=0}^{t+1} C_s = \left[\prod_{s=1}^{t+1} p(a_{s+1}, x_{2,s+1} | \mathcal{F}_s)\right] p(a_0, x_{2,0} \,|\, \theta)$$

We Show a More General Result

Given draws $\tilde{x}_{1,0:t}^{(k)}$ and weights $\tilde{w}_t^{(k)}$, $k = 1, \ldots, N$, that satisfy

$$\int g(x_{1,0:t})\, dP(x_{1,0:t}|\mathcal{F}_t) = \tilde{\mathcal{E}}_{1,0:t}\left\{\mathcal{E}\left[\sum_{k=1}^{N} \tilde{w}_t^{(k)} g(\tilde{x}_{1,0:t}^{(k)}) \mid \mathcal{F}_t\right]\right\}$$

for integrable $g(x_{1t})$, we seek to generate draws $\tilde{x}_{1,t+1}^{(k)}$ and compute weights $\tilde{w}_{t+1}^{(k)}$ that satisfy

$$\int\int g(x_{1,0:t}, x_{1,t+1})\, dP(x_{1,0:t}, x_{1,t+1}|\mathcal{F}_{t+1})$$

$$= \tilde{\mathcal{E}}_{1,t+1}\tilde{\mathcal{E}}_{1,0:t}\left\{\mathcal{E}\left[\sum_{k=1}^{N} \tilde{w}_{t+1}^{(k)}\, g(\tilde{x}_{1,0:t}^{(k)}, \tilde{x}_{1,t+1}^{(k)}) \mid \mathcal{F}_{t+1}\right]\right\}$$

for integrable $g(x_{1,0:t}, x_{1t+1})$.

Bayes Theorem

$$p(x_{1,0:t}, x_{1,t+1}|a_{t+1}, x_{2,t+1}, \mathcal{F}_t) = \frac{p(a_{t+1}, x_{2,t+1}, x_{1,0:t}, x_{1,t+1}|\mathcal{F}_t)}{p(a_{t+1}, x_{2,t+1}|\mathcal{F}_t)}.$$

However

$$\begin{aligned} p(x_{1,0:t}, x_{1,t+1}|a_{t+1}, x_{2,t+1}, \mathcal{F}_t) & \\ &= p(x_{1,0:t}, x_{1,t+1}|\mathcal{F}_{t+1}) \\ p(a_{t+1}, x_{2,t+1}, x_{1,0:t}, x_{1,t+1}|\mathcal{F}_t) & \\ &= p(a_{t+1}, x_{2,t+1}|x_{1,0:t}, x_{1,t+1}, \mathcal{F}_t)p(x_{1,t+1}|x_{1,0:t}, \mathcal{F}_t)p(x_{1,0:t}|\mathcal{F}_t) \\ C_{t+1} &= p(a_{t+1}, x_{2,t+1}|\mathcal{F}_t) \end{aligned}$$

Proof

Given $\tilde{w}_t^{(k)}$ and $\tilde{x}_t$, draw $\tilde{x}_{1,t+1}^{(k)}$ from $p(x_{1,t+1}|x_{1,0:t}^{(k)}, \mathcal{F}_t)$ and define

$$\tilde{w}_{t+1}^{(k)} = \frac{p(a_{t+1}, x_{2,t+1}|\tilde{x}_{1,0:t}^{(k)}, \tilde{x}_{1,t+1}^{(k)}, \mathcal{F}_t)}{p(a_{t+1}, x_{2,t+1}|\mathcal{F}_t)}\, \tilde{w}_t^{(k)}.$$

$$\iint g(x_{1,0:t}, x_{1,t+1})\, dP(x_{1,0:t}, x_{1,t+1}|\mathcal{F}_{t+1})$$

$$= \iint g(x_{1,0:t}, x_{1,t+1}) \frac{p(a_{t+1}, x_{2,t+1}|x_{1,0:t}, x_{1,t+1}, \mathcal{F}_t)}{p(a_{t+1}, x_{2,t+1}|\mathcal{F}_t)} p(x_{1,t+1}|x_{1,0:t}, \mathcal{F}_t)$$
$$\times d\lambda(x_{1,t+1}) dP(x_{1,0:t}|\mathcal{F}_t)$$

$$= \tilde{\mathcal{E}}_{1,0:t}\, \mathcal{E}\left[\int \sum_{k=1}^{N} g(\tilde{x}_{1,0:t}^{(k)}, x_{1,t+1})\, \tilde{w}_{t+1}^{(k)}\, p(x_{1,t+1}|\tilde{x}_{1,0:t}^{(k)}, \mathcal{F}_t)\, d\lambda(x_{1,t+1}) \mid \mathcal{F}_t\right]$$

$$= \tilde{\mathcal{E}}_{1,t+1} \tilde{\mathcal{E}}_{1,0:t}\, \mathcal{E}\left[\sum_{k=1}^{N} g(\tilde{x}_{1,0:t}^{(k)}, \tilde{x}_{1,t+1}^{(k)})\, \tilde{w}_{t+1}^{(k)} \mid \mathcal{F}_{t+1}\right]$$

Second to last $\tilde{w}_{t+1}^{(k)}$ is actually $\tilde{w}_{t+1}^{(k)}\Big|_{\tilde{x}_{t+1}^{(k)}=x_{t+1}}$

Specialization

- Put $1 = g(x_{1,0:t}) = g(x_{1,0:t}, x_{1,t+1})$.

- Realize that the denominator of $\tilde{w}_{t+1}^{(k)}$ is C_{t+1}.

- Algebra to express the numerator of $\tilde{w}_{t+1}^{(k)}$ in terms of problem primitives.

- Show that resampling does not affect the result as long as scale is preserved.

- Use a telescoping argument to show that weights can be normalized to sum to one at a certain point in the algorithm.

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Design – 1

- Three firms, time increment one year.
  - ▷ β is 20% internal rate of return
  - ▷ μ_c and μ_r imply 30% profit margin, persistent ρ_c
  - ▷ κ_a is a 20% hit to margin with ρ_a at 6 mo. half life.
  - ▷ σ_c and σ_r chosen to prevent monopoly
  - ▷ Outcome uncertainty $1 - p_a$ is 5% (from an application).

- Simulated from fully rational model.

$$
\begin{aligned}
\theta &= (\mu_c, \rho_c, \sigma_c, \mu_r, \sigma_r, \rho_a, \kappa_a, \beta, p_a) \\
&= (9.7, 0.9, 0.1, 10.0, 2.0, 0.5, 0.2, 0.83, 0.95) \\
T_0 &= 160,\ \text{sm} : T = 40,\ \text{md} : T = 120,\ \text{lg} : T = 360
\end{aligned}
$$

Design – 2

1. Fit fully rational model, blind proposal, and multinomial resampling.

2. Fit boundedly rational model, blind proposal, and multinomial resampling.

3. Fit fully rational model, adaptive proposal, and multinomial resampling.

4. Fit fully rational model, adaptive proposal, and systematic resampling.

Results – 1

- A large sample size is better. In Tables 3 through 6 the estimates shown in the columns labeled "lg" would not give misleading results in an application.

- In smaller sample sizes the specification error caused by fitting the boundedly rational model to data generated by the fully rational model can be serious: compare columns "sm" and "md" in Tables 3 and 4. The saving in computational time is about 10% relative to the fully rational model so there seems to be no point to using the boundedly rational model unless that is what firms are actually doing, which they are not in this instance.

Results – 2

- Constraining β is beneficial: compare Figures 1 and 2. The constraint reduces the bimodality of the marginal posterior distribution of σ_r and pushes all histograms closer to unimodality.

- Constraining p_a is irrelevant except for a small savings in computational cost: compare columns "β" and "β & p_a" in Tables 3 through 6.

- Improvements to the particle filter are helpful. In particular, an adaptive proposal is better than a blind proposal; compare Tables 3 and 5 and compare Figures 3 and 4. Systematic resampling is better than multinomial resampling; compare Tables 5 and 6.

Table 3. Fully Rational Estimates, Blind Proposal, Multinomial Resampling

Parameter		Unconstrained			Constrained					
					β			β & p_a		
	value	sm	md	lg	sm	md	lg	sm	md	lg
μ_c	9.70	10.10	9.72	9.68	9.94	9.67	9.68	9.86	9.72	9.68
		(0.15)	(0.12)	(0.06)	(0.19)	(0.11)	(0.06)	(0.18)	(0.12)	(0.06)
ρ_c	0.90	0.58	0.86	0.92	0.69	0.92	0.91	0.69	0.85	0.91
		(0.25)	(0.09)	(0.03)	(0.26)	(0.05)	(0.03)	(0.25)	(0.11)	(0.03)
σ_c	0.10	0.16	0.09	0.09	0.17	0.08	0.10	0.15	0.09	0.10
		(0.05)	(0.03)	(0.01)	(0.06)	(0.03)	(0.01)	(0.07)	(0.03)	(0.01)
μ_r	10.00	9.87	9.98	9.96	9.88	9.99	9.98	9.84	9.99	9.99
		(0.10)	(0.03)	(0.02)	(0.10)	(0.03)	(0.02)	(0.13)	(0.06)	(0.02)
σ_r	2.00	1.95	1.97	1.98	2.02	2.00	2.02	2.04	2.00	2.03
		(0.09)	(0.05)	(0.01)	(0.08)	(0.02)	(0.02)	(0.10)	(0.03)	(0.01)
ρ_a	0.50	0.76	0.56	0.58	0.59	0.57	0.56	0.76	0.57	0.52
		(0.09)	(0.07)	(0.06)	(0.22)	(0.09)	(0.05)	(0.10)	(0.07)	(0.04)
κ_a	0.20	0.04	0.24	0.19	0.15	0.26	0.20	0.14	0.22	0.22
		(0.05)	(0.05)	(0.02)	(0.07)	(0.05)	(0.03)	(0.06)	(0.06)	(0.03)
β	0.83	0.90	0.95	0.87	0.83	0.83	0.83	0.83	0.83	0.83
		(0.07)	(0.04)	(0.04)						
p_a	0.95	0.97	0.94	0.95	0.96	0.94	0.95	0.95	0.95	0.95
		(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)			

Table 4. Boundedly Rational Estimates, Blind Proposal, Multinomial Resampling

Parameter		Unconstrained			Constrained					
					β			β & p_a		
	value	sm	md	lg	sm	md	lg	sm	md	lg
μ_c	9.70	10.06	9.71	9.69	9.71	9.48	9.64	9.90	9.57	9.66
		(0.18)	(0.10)	(0.06)	(0.18)	(0.13)	(0.06)	(0.23)	(0.14)	(0.05)
ρ_c	0.90	0.80	0.92	0.91	0.83	0.95	0.90	0.73	0.94	0.92
		(0.13)	(0.03)	(0.02)	(0.13)	(0.04)	(0.03)	(0.20)	(0.03)	(0.03)
σ_c	0.10	0.31	0.08	0.09	0.13	0.06	0.09	0.13	0.07	0.09
		(0.13)	(0.02)	(0.01)	(0.06)	(0.02)	(0.02)	(0.05)	(0.02)	(0.01)
μ_r	10.00	9.84	9.96	9.96	9.91	9.99	9.92	9.82	10.00	9.94
		(0.07)	(0.02)	(0.03)	(0.08)	(0.02)	(0.04)	(0.14)	(0.02)	(0.04)
σ_r	2.00	1.91	1.95	1.99	1.93	1.96	1.99	2.00	2.01	2.00
		(0.09)	(0.04)	(0.03)	(0.05)	(0.05)	(0.02)	(0.09)	(0.05)	(0.02)
ρ_a	0.50	0.22	0.47	0.52	0.72	0.56	0.55	0.78	0.55	0.57
		(0.15)	(0.13)	(0.06)	(0.15)	(0.06)	(0.07)	(0.06)	(0.07)	(0.05)
κ_a	0.20	0.01	0.25	0.19	0.19	0.36	0.20	0.10	0.32	0.19
		(0.14)	(0.05)	(0.02)	(0.07)	(0.08)	(0.03)	(0.10)	(0.07)	(0.02)
β	0.83	0.61	0.95	0.85	0.83	0.83	0.83	0.83	0.83	0.83
		(0.28)	(0.04)	(0.06)						
p_a	0.95	0.97	0.93	0.95	0.97	0.94	0.95	0.95	0.95	0.95
		(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)			

Table 5. Fully Rational Estimates, Adaptive Proposal, Multinomial Resampling

					Constrained					
Parameter		Unconstrained			β			β & p_a		
	value	sm	md	lg	sm	md	lg	sm	md	lg
μ_c	9.70	10.00	9.82	9.77	9.93	9.74	9.70	9.85	9.73	9.65
		(0.24)	(0.07)	(0.05)	(0.12)	(0.07)	(0.06)	(0.15)	(0.09)	(0.05)
ρ_c	0.90	0.95	0.85	0.87	0.87	0.92	0.93	0.87	0.92	0.94
		(0.03)	(0.07)	(0.05)	(0.08)	(0.04)	(0.03)	(0.09)	(0.04)	(0.02)
σ_c	0.10	0.14	0.09	0.10	0.12	0.08	0.08	0.12	0.09	0.08
		(0.02)	(0.02)	(0.01)	(0.04)	(0.02)	(0.01)	(0.04)	(0.03)	(0.01)
μ_r	10.00	9.93	10.00	10.01	10.00	9.99	9.97	9.94	9.96	9.96
		(0.06)	(0.02)	(0.01)	(0.05)	(0.02)	(0.02)	(0.07)	(0.03)	(0.03)
σ_r	2.00	1.93	1.98	1.99	2.01	1.98	2.00	2.03	1.97	1.99
		(0.10)	(0.02)	(0.02)	(0.09)	(0.01)	(0.01)	(0.09)	(0.02)	(0.02)
ρ_a	0.50	-0.11	0.51	0.47	0.56	0.59	0.57	0.47	0.51	0.61
		(0.21)	(0.09)	(0.06)	(0.17)	(0.06)	(0.06)	(0.20)	(0.07)	(0.05)
κ_a	0.20	0.19	0.20	0.17	0.17	0.21	0.18	0.24	0.20	0.19
		(0.02)	(0.03)	(0.02)	(0.06)	(0.02)	(0.02)	(0.03)	(0.02)	(0.02)
β	0.83	0.87	0.95	0.92	0.83	0.83	0.83	0.83	0.83	0.83
		(0.10)	(0.03)	(0.04)						
p_a	0.95	0.95	0.94	0.95	0.96	0.95	0.95	0.95	0.95	0.95
		(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)			

Table 6. Fully Rational Estimates, Adaptive Proposal, Systematic Resampling

Parameter		Unconstrained			Constrained					
					β			β & p_a		
	value	sm	md	lg	sm	md	lg	sm	md	lg
μ_c	9.70	9.87	9.82	9.72	9.81	9.78	9.68	9.78	9.76	9.65
		(0.24)	(0.07)	(0.05)	(0.12)	(0.07)	(0.06)	(0.15)	(0.09)	(0.05)
ρ_c	0.90	0.77	0.82	0.91	0.93	0.94	0.94	0.86	0.92	0.94
		(0.03)	(0.07)	(0.05)	(0.08)	(0.04)	(0.03)	(0.09)	(0.04)	(0.02)
σ_c	0.10	0.14	0.10	0.09	0.14	0.08	0.08	0.11	0.08	0.08
		(0.02)	(0.02)	(0.01)	(0.04)	(0.02)	(0.01)	(0.04)	(0.03)	(0.01)
μ_r	10.00	10.05	10.00	9.97	9.95	9.96	9.94	9.78	9.95	9.96
		(0.06)	(0.02)	(0.01)	(0.05)	(0.02)	(0.02)	(0.07)	(0.03)	(0.03)
σ_r	2.00	1.94	1.99	1.99	1.93	1.97	2.01	2.07	1.98	1.97
		(0.10)	(0.02)	(0.02)	(0.09)	(0.01)	(0.01)	(0.09)	(0.02)	(0.02)
ρ_a	0.50	0.61	0.53	0.56	0.41	0.36	0.61	0.71	0.58	0.64
		(0.21)	(0.09)	(0.06)	(0.17)	(0.06)	(0.06)	(0.20)	(0.07)	(0.05)
κ_a	0.20	0.21	0.22	0.18	0.20	0.18	0.18	0.17	0.19	0.18
		(0.02)	(0.03)	(0.02)	(0.06)	(0.02)	(0.02)	(0.03)	(0.02)	(0.02)
β	0.83	0.93	0.96	0.90	0.83	0.83	0.83	0.83	0.83	0.83
		(0.10)	(0.03)	(0.04)						
p_a	0.95	0.96	0.94	0.95	0.95	0.93	0.95	0.95	0.95	0.95
		(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)			

Figure 1. Fully Rational Distributions, Unconstrained, Blind Proposal.

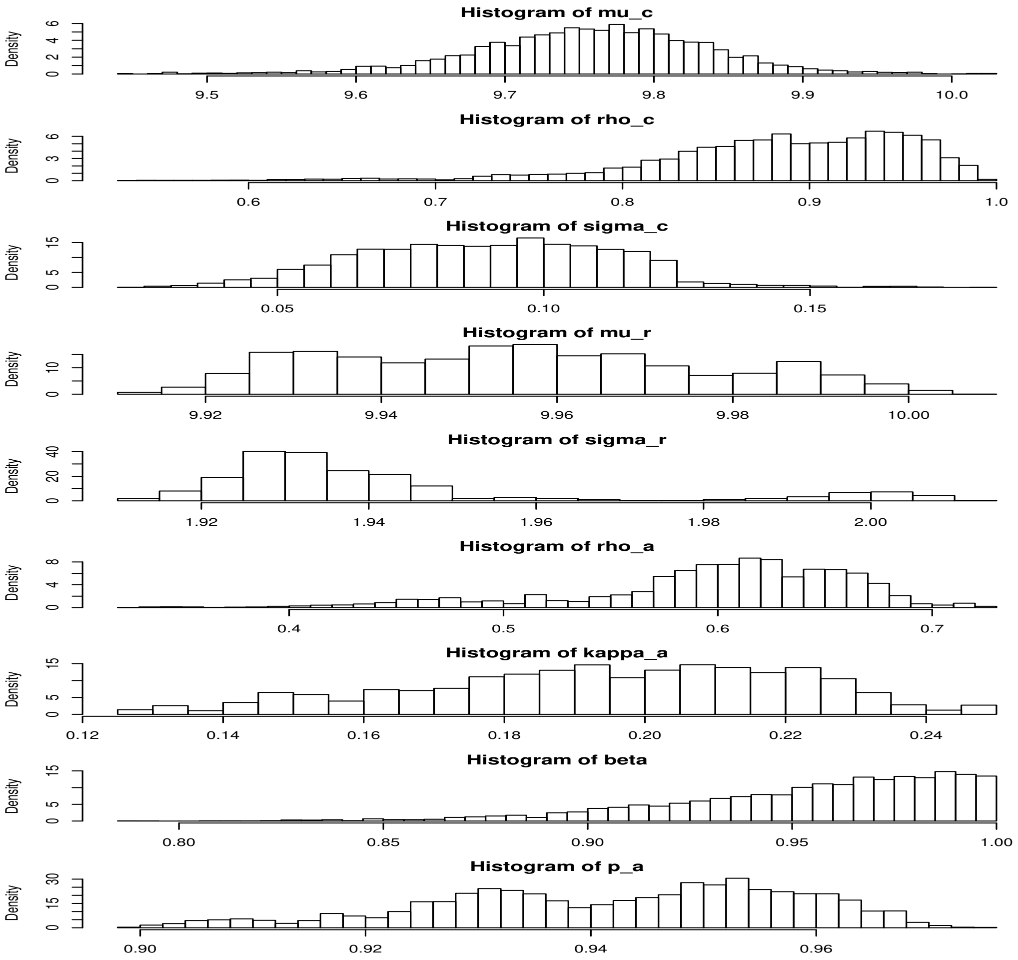


Figure 2. Fully Rational Distributions, β Constrained, Blind Proposal.

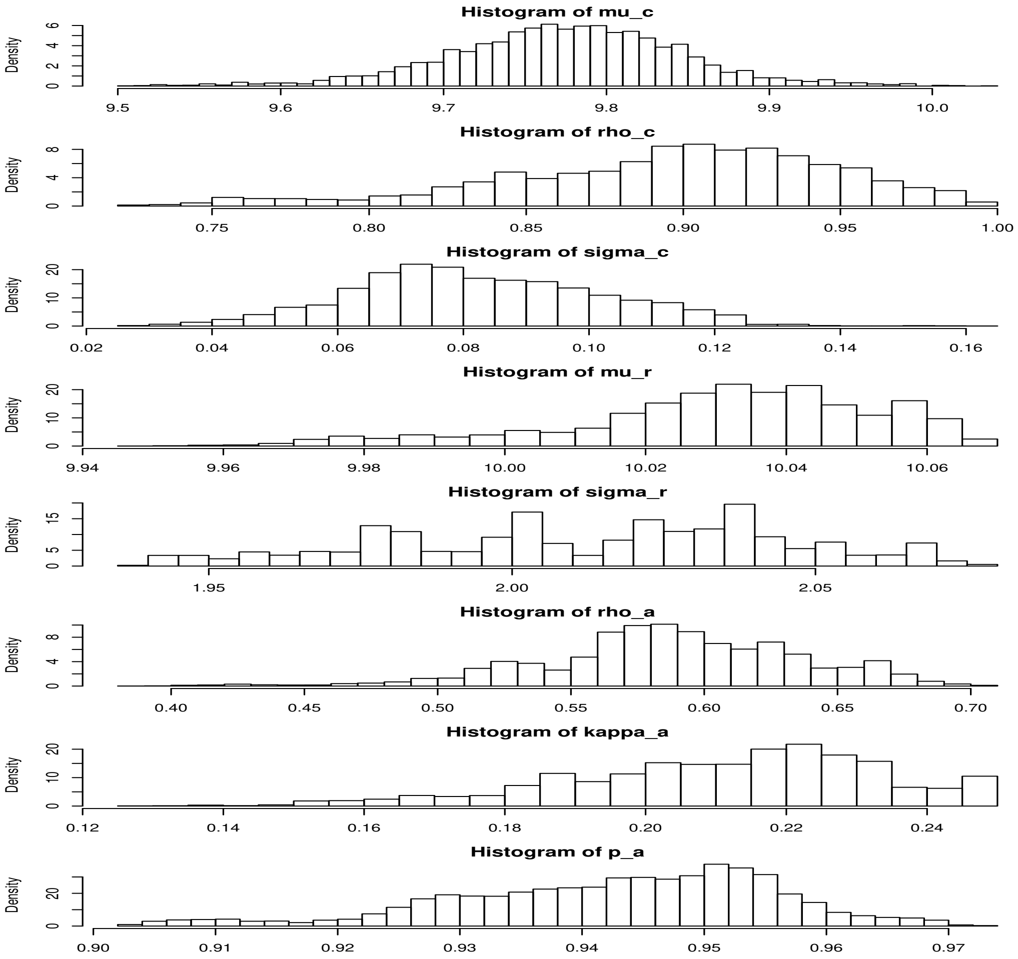
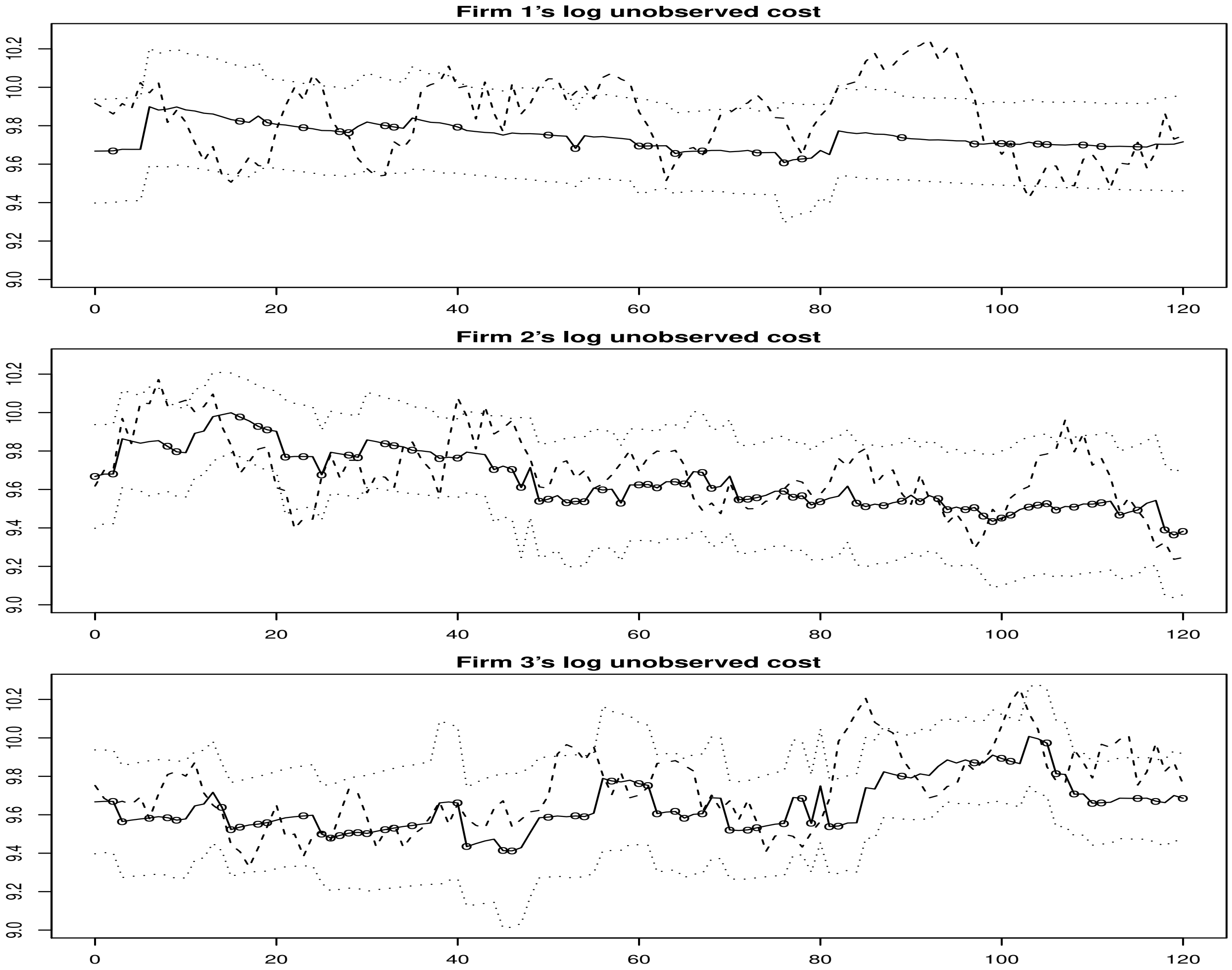
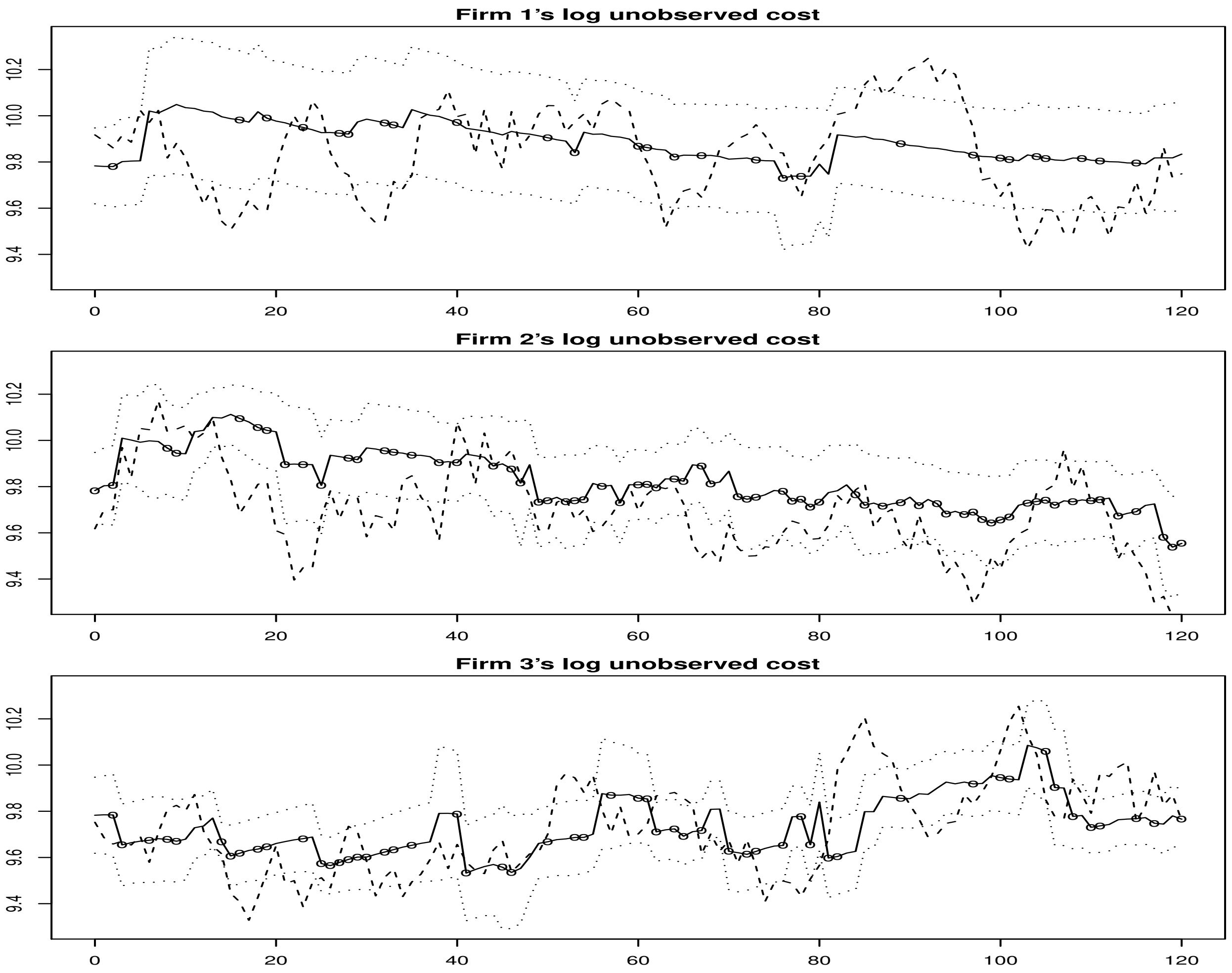


Figure 3. Fully Rational Cost Estimates, β Constrained, Blind Proposal.



Circles indicate entry. Dashed line is true unobserved cost. The solid line is the average of β constrained estimates over all MCMC repetitions, with a stride of 25. The dotted line is $\pm$ 1.96 standard deviations about solid line. The sum of the norms of the difference between the solid and dashed lines is 0.186146.

Figure 4. Fully Rational Cost Estimates, β Constrained, Adaptive Proposal.



Circles indicate entry. Dashed line is true unobserved cost. The solid line is the average of β constrained estimates over all MCMC repetitions, with a stride of 25. The dotted line is $\pm$ 1.96 standard deviations about solid line. The sum of the norms of the difference between the solid and dashed lines is 0.169411.